Imperial College London

Lecture 5

Nodal Analysis

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In this lecture, we will consider how to analyse an electrical circuit by applying KVL and KCL. As a result, we can predict the voltages and currents around an electrical circuit. This is a short lecture, but the content is important and fundamental to understanding the basic of electrical engineering as applied to Design Engineering.

Aim of Nodal Analysis

- The aim of nodal analysis is to determine the voltage at each node relative to the **reference node** (or ground). Once you have done this you can easily work out anything else you need.
- There are two ways to do this:
  (1) Nodal Analysis - systematic; always works
  (2) Circuit Manipulation - ad hoc; but can be less work and clearer

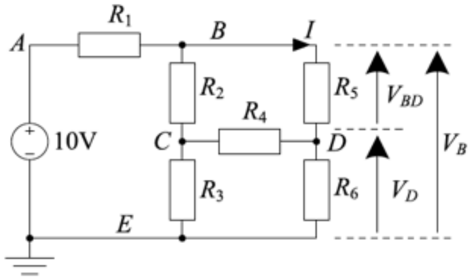
Reminders:

- A node is all the points in a circuit that are directly interconnected.
- We assume the interconnections have zero resistance so all points within a node have the **same voltage**. Five nodes: $A, \ldots\ldots\ldots, E$.

Ohm's Law: $V_{BD} = I R_5$

KVL: $V_{BD} = V_B - V_D$

KCL: Total current exiting any closed region is zero.



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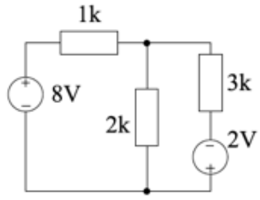
In this lecture, you will learn about a technical known as "**nodal analysis**". If you want to calculate (i.e. predict) voltages and currents in different part of an electrical circuit, you could manipulate the circuits through simplification via substituting components with their equivalent etc. However such approach does not always work.

Nodal analysis is a **systematic way** of analysing a circuit using KCL or KVL, and it always works.

You need to remember what are nodes, KCL, KVL, Ohm's Law and that all interconnections (nodes) have zero resistance.

Nodal Analysis Stage 1: Label Nodes

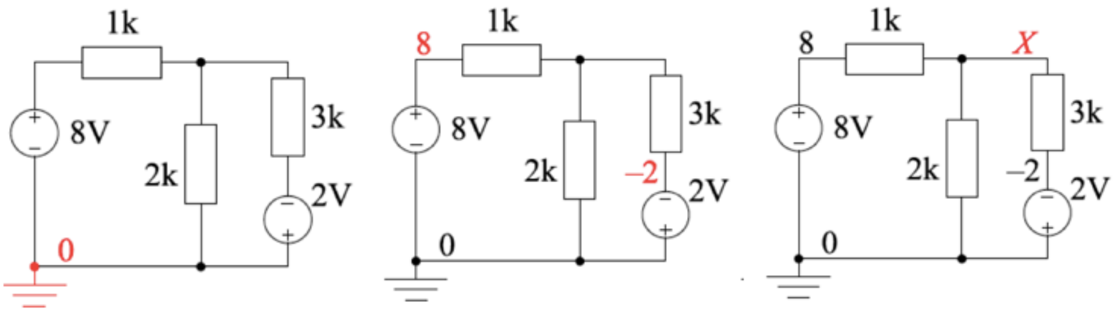
- To find the voltage at each node, the first step is to label each node with its voltage as follows:



(1) Pick any node as the voltage reference. Label its voltage as $0V$.

(2) If any fixed voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.

(3) Pick an unlabelled node and label it with $X, Y, \ldots$, then go back to step (2) until all nodes are labelled.



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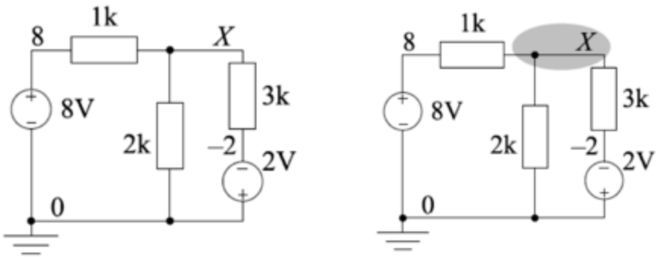
Let us consider a simple circuit as shown here. We need to find voltages at all nodes.

First we pick a node to be reference (or Ground). This is our 0v.

Identify all fixed voltage sources and label these with the voltage values. What remains is one node unlabelled, which we label as X.

Nodal Analysis Stage 2: KCL Equations

- The second step is to write down a KCL equation for each node labelled with a variable by setting the total current flowing out of the node to zero.
- For a circuit with N nodes and S voltage sources you will have $N - S - 1$ simultaneous equations to solve.



- We only have one variable:

$$\frac{X-8}{1\,\mathrm{k}} + \frac{X-0}{2\,\mathrm{k}} + \frac{X-(-2)}{3\,\mathrm{k}} = 0 \quad \Rightarrow \quad (6X - 48) + 3X + (2X + 4) = 0$$

$$11X = 44 \quad \Rightarrow \quad X = 4$$

Numerator for a resistor is always of the form $X - V_N$ where V_N is the voltage on the other side of the resistor.

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Next we apply KCL to **each** node – i.e. all current at a node sum to zero. A circuit with N nodes and S sources, there will be N-S-1 different unlabelled nodes (S source nodes labelled with fixed voltages, and the 1 is for the 0v reference).

For this circuit we only have one unknown X. Sum all current flowing OUT of X. And we get X (i.e. voltage at node X relative to 0 node) = 4V.

Current Sources

- Current sources cause no problems.

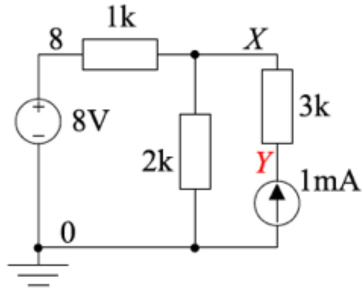
  (1) Pick reference node.

  (2) Label nodes: 8, X and Y .

  (3) Write equations

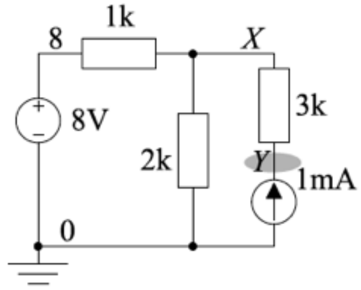
$$\frac{X-8}{1} + \frac{X}{2} + \frac{X-Y}{3} = 0$$

$$\frac{Y-X}{3} + (-1) = 0$$



Ohm’s law works OK if all resistors are in k and all currents in mA.

(4) Solve the equations: X = 6, Y = 9



Let us replace one of the voltage source with a current source. We can use the same method to analyse this circuit.

Pick the reference node and label this.

Label the other nodes 8, X and Y.

Now use KCL at X and Y, and we get the equations shown.

Since all resistors are in kilo ohms, and all currents are in mA, with Ohm’s Law, V=IR, therefore k x m = 1. We can ignore the multipliers – it will work.

Solving the simultaneous equation gives us X = 6 and Y = 9 (both in V).

Floating Voltage Sources

- ***Floating voltage sources*** have neither end connected to a known fixed voltage. We have to change how we form the KCL equations slightly.

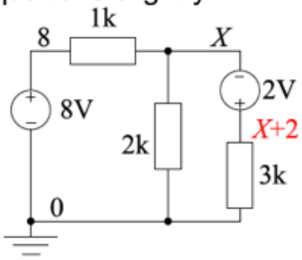
  (1) Pick reference node.

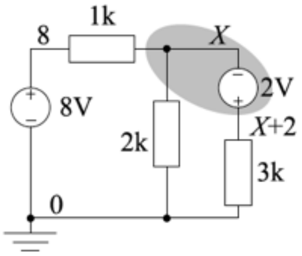
  (2) Label nodes: 8, X and $X+2$ since it is joined to X via a voltage source.

  (3) Write KCL equations but count all the nodes connected via floating voltage sources as a single "*super-node*" giving one equation

$$\frac{X-8}{1}+\frac{X}{2}+\frac{(X+2)-0}{3}=0$$

  (4) Solve the equations: X = 4





Ohm's law always involves the difference between the voltages at either end of a resistor. (Obvious but easily forgotten)

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So far the voltage sources have been connected to 0V reference at one end. That makes the calculation easy because both end of the source have known voltages.

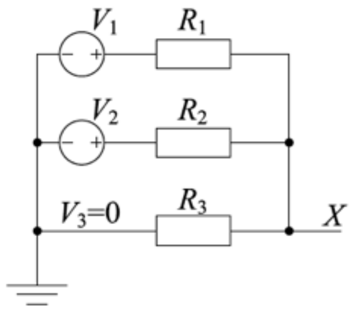
If the voltage source do not have either end connected to a known voltage, it is called a floating voltage source.

Handling such a source is easy. Simply label one end as an unknown voltage, say X, and the other end is related to X. In the example shown here, the negative end of the source is labelled X. The positive end is simply X+2.

When apply KCL to analyse the circuit, instead of summing current at a node, apply KCL to the entire source shown in gray. This region is treated as a super node. Applying this method results in having only one equation for this circuit.

Weighted Average Circuit

- A very useful sub-circuit that calculates the weighted average of any number of voltages.



- KCL equation for node X:

$$\frac{X-V_1}{R_1} + \frac{X-V_2}{R_2} + \frac{X-V_3}{R_3} = 0$$

- Still works if $V_3 = 0$.

- Or using conductances:

$$(X - V_1)G_1 + (X - V_2)G_2 + (X - V_3)G_3 = 0$$

$$X(G_1 + G_2 + G_3) = V_1G_1 + V_2G_2 + V_3G_3$$

$$X = \frac{V_1G_1+V_2G_2+V_3G_3}{G_1+G_2+G_3} = \frac{\sum V_iG_i}{\sum G_i}$$

Voltage X is the average of V1, V2, V3 weighted by the conductances.

Consider another example circuit. This one is very useful because it produces a WEIGHTED AVERAGE of a number of voltage sources.

The algebraic manipulation produces an output X, which is the average of V1, V2 and V3 weighted by the conductance of each branch.

$$X = V_1 \frac{G_1}{G_1 + G_2 + G_3} + V_2 \frac{G_2}{G_1 + G_2 + G_3} + V_3 \frac{G_4}{G_1 + G_2 + G_3}$$

Simple Digital to Analogue Converter

A 3-bit binary number, b, has bit-weights of 4, 2 and 1. Thus 110 has a value 6 in decimal. If we label the bits $b_2b_1b_0$, then $b = 4b_2 + 2b_1 + b_0$.

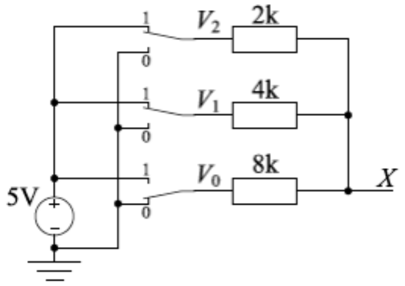
We use $b_2b_1b_0$ to control the switches which determine whether $V_i = 5\,\mathrm{V}$ or $V_i = 0\,\mathrm{V}$. Thus $V_i = 5b_i$. Switches shown for $b = 6$.

$$X = \frac{\frac{1}{2}V_2+\frac{1}{4}V_1+\frac{1}{8}V_0}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$$

$$= \frac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$

but $V_i = 5 \times b_i$ since it connects to either $0\,\mathrm{V}$ or $5\,\mathrm{V}$

$$= \frac{5}{7}\left(4b_2 + 2b_1 + b_0\right) = \frac{5}{7}b$$



$$G_2 = \frac{1}{R_2} = \frac{1}{2\,k} = \frac{1}{2}\,\mathrm{mS}, \ \ldots$$

So we have made a circuit in which X is proportional to a binary number b.

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This simple circuit can be used to produce a simple digital-to-analogue converter (DAC). Consider the circuit shown above. The three digital switches are controlled by three binary bits: $b_2b_1b_0$. When the digital bit is high b_i (‘1’), the switch SW_i is connected to 5v voltage source, otherwise it is connect to ground.

Using the results from last slide, we can see that the voltage X is given by:

$$X = \frac{\frac{1}{2}V_2+\frac{1}{4}V_1+\frac{1}{8}V_0}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = \frac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$

Therefore X is proportional to the digital value of the binary number $b_2b_1b_0$.

Dependent Voltage Sources

- A *dependent* voltage or current source is one whose value is determined by voltages or currents elsewhere in the circuit. These are most commonly used when modelling the behaviour of transistors or op-amps. Each dependent source has a *defining equation*.
- In this circuit: $I_S = 0.2W$ mA where W is in volts.
  - (1) Pick reference node.
  - (2) Label nodes: $0,\ U,\ X$ and Y.
  - (3) Write equation for the dependent source, I_S, in terms of node voltages:

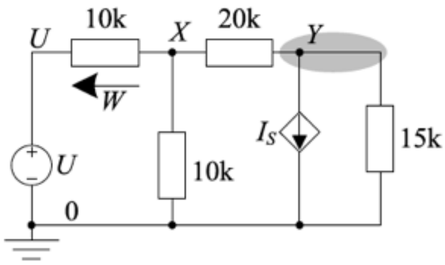
    $$I_S = 0.2\,(U - X)$$

  - (4) Write KCL equations:

    $$\frac{X-U}{10} + \frac{X}{10} + \frac{X-Y}{20} = 0 \qquad \frac{Y-X}{20} + I_S + \frac{Y}{15} = 0$$

  - (5) Solve all three equations to find X, Y and IS in terms of U:

    $$X = 0.1U,\ Y = -1.5U,\ I_S = 0.18U$$



Note that the value of U is assumed to be known.

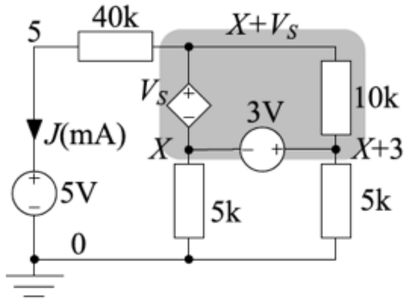
Here is a new component known as **DEPENDENT source** (voltage or current). Instead of having a fixed voltage or current value, its output value is determined by voltage or current **elsewhere in a circuit**. This is often used to model the behaviour of more complicated circuits such as an amplifier or even an entire system. We will come back to modeling and amplification in a later lecture.

Each dependent source has a defining equation which provide the relationship between the source value and the causal parameter. In the circuit shown, the current source IS is controlled by the voltage W across the 10k resistor.

Applying KCL at X and Y. We have three equations (the defining equation for the current source and two KCL equations) and four unknowns: U, IS, X and Y. Solving the three equations provides a solution in terms of U, the voltage source value, which is assumed to be known.

Dependent Voltage Sources

- The value of the highlighted dependent voltage source is $V_S = 10J$ Volts where J is the indicated current in mA.



(1) Pick reference node.

(2) Label nodes: 0, 5, X, $X+3$ and $X+V_S$.

(3) Write equation for the dependent source, V_S, in terms of node voltages:

$$V_S = 10J = 10 \times \frac{X+V_S-5}{40} \Rightarrow 3V_S = X-5$$

(4) Write KCL equations: all nodes connected by floating voltage sources and all components connecting these nodes are in the same "super-node"

$$\frac{X+V_S-5}{40} + \frac{X}{5} + \frac{X+3}{5} = 0$$

(5) Solve the two equations: $X = -1$ and $V_S = -2$

Here is a more complex example with dependent voltage sources. The dependent voltage source V_S is governed by the current J. Remember, we generally use units of mA for currents, k ohms for resistors and V for voltages.

First we write an equation for the dependent voltage source in terms of node voltage in step 3).

Next we create a "**super-node**" that includes the floating voltage source AND all components connected to the source V_S. The two nodes of the source are X and $X+V_S$. Here we include the 3V fixed source and the 10k resistor because including these components do not add any new unknown variables.

Now apply KCL to this super-node (or region) and produces one equation in terms of X and V_S only. Two equations, two unknowns, we have our solution.

Universal Nodal Analysis Algorithm

① Pick any node as the voltage reference. Label its voltage as 0 V. Label any dependent sources with VS, IS, ...

② If any voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.

③ Pick an unlabelled node and label it with X, Y, . . ., then loop back to step (2) until all nodes are labelled.

④ For each dependent source, write down an equation that expresses its value in terms of other node voltages.

⑤ Write down a KCL equation for each "normal" node (i.e. one that is not connected to a floating voltage source).

⑥ Write down a KCL equation for each "super-node". A super-node consists of a set of nodes that are joined by floating voltage sources and includes any other components joining these nodes.

⑦ Solve the set of simultaneous equations that you have written down.

Let me recap. Here are the steps in nodal analysis. You would have realized that we have only used KCL in this example. You could have used KVL (summing voltages in a closed loop equals zero) and have the same results. However, generally you will find that using KCL and summing current is slightly easier to compute.

Doing nodal analysis is actually easier than it first appears. However it requires practice. I have provided a large number of example questions in the tutorial problem sheet 2 for you to try. The trick is to "*see*" the circuit in a way that makes the circuit as simple as possible. Then work out how many unknowns there are, and which are the simplest equations to produce in order to solve the unknowns.

Summary

- Nodal Analysis
  - Simple Circuits (no floating or dependent voltage sources)
  - Floating Voltage Sources
    - use supernodes: all the nodes connected by floating voltage sources (independent or dependent)
  - Dependent Voltage and Current Sources
    - Label each source with a variable
    - Write extra equations expressing the source values in terms of node voltages
    - Write down the KCL equations as before
- Mesh Analysis (in most textbooks)
  - Alternative to nodal analysis but doesn't work for all circuits
  - No significant benefits $\Rightarrow$ ignore it